Using weak impulses to suppress traveling waves in excitable media

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(Received 9 February 1999)

Here we propose mechanisms for suppressing non-steady-state motions—propagating pulses, spiral waves, spiral-wave chaos—in excitable media. Our approach is based on two points: (1) excitable media are multistable; and (2) traveling waves in excitable media can be separated into fast and slow motions, which can be considered independently. We show that weak impulses can be used to change the values of the slow variable at the front and back of a traveling wave, which leads to wave front and wave back velocities that are different from each other. This effect can destabilize the traveling wave, resulting in a transition to the rest state. [S1063-651X(99)00206-8]

PACS number(s): 82.20.Wt, 05.45.−a

Excitable media are widespread in physics, chemistry, and biology [1]. In some applications, propagating waves in excitable media are viewed as an undesirable effect. For example, the spontaneous breakup of a spiral wave into several waves and their subsequent multiplication can lead to spatiotemporal chaos [2]. Such dynamics have been implicated as a possible mechanism for the onset of ventricular fibrillation in the heart [3], which is one of the leading causes of death in industrialized countries. Accordingly, there is a need to develop effective methods for suppressing traveling waves in excitable media [4–13].

Here we present methods that utilize *weak impulses* to annihilate non-steady-state motions, such as propagating pulses, spiral waves, and spiral-wave chaos, in excitable media. Our approach is based on two points: (1) excitable media are multistable, that is, there can exist two or more linearly stable states, which can be nonlinearly unstable (i.e., relatively large perturbations can change the system's attractor) [14]; and (2) traveling waves in excitable media can be separated into fast and slow motions, which can be considered separately [15].

We consider the following model that describes an excitable medium in terms of reaction-diffusion equations:

$$\frac{\partial u}{\partial t}=f(u,v)+D\nabla^2 u, \quad \frac{\partial v}{\partial t}=\varepsilon g(u,v), \tag{1}$$

where we take the functions $f(u,v)$ and $g(u,v)$ to have the following forms (as in the FitzHugh-Nagumo equations): $f(u,v)=u-u^3/3-v$, $g(u,v)=u-\gamma v+\beta$.

We only consider small values for the relaxation parameter ε, which controls the spatiotemporal scale separation, making u and v in Eqs. (1) fast and slow variables, respectively. Parameter β defines the asymmetry between the excitation and recovery for each element, γ characterizes the dissipation of the slow variable v, and D is the diffusion coefficient for the fast variable. The model describing the partial dynamics of the medium [in Eqs. (1), $D=0$] has a steady state at the intersection of the nullclines $f(u,v)=0$ and $g(u,v)=0$. The coordinates (u^0,v^0), and stability of the steady state are defined by the values of β and γ. If the steady state of a partial cell lies in the interval $u<-1$ or $u>1$, then it is stable. We consider the case where $u^0<-1$.

We consider parameter values $\varepsilon=0.005$, $\beta=0.7$, $\gamma=0.5$, and $D=1.0$. At these values, the excitable medium described by Eqs. (1) is multistable, i.e., there exists more than one attractor.

In the one-dimensional case for the above system, $\nabla^2 u=\partial^2 u/\partial x^2$ and $x\in[0,L]$, where $L(=50$ and $100)$ is the size of the medium. A propagating pulse can exist in this system (see Fig. 1) with periodic boundary conditions $u(x+L)=u(x)$, together with the spatially homogeneous steady state $u(x,t)=u^0$ and $v(x,t)=v^0$. We consider the conditions under which a weak impulse can be used to suppress the propagating pulse. A weak impulse acting on the whole medium can be included as an external force $e(t)$ in the second equation of system (1):

$$\frac{\partial u}{\partial t}=f(u,v)+D\frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial v}{\partial t}=\varepsilon g(u,v)+e, \tag{2}$$

where

$$e\equiv e(t)=\begin{cases} E_0, & t\in\Delta t \\ 0, & t\notin\Delta t, \end{cases} \tag{3}$$

where E_0 is the amplitude of the impulse, and the value of the interval Δt determines the duration of the impulse.

The characteristic time of evolution of the slow variable $v(x,t)$ is $\tau=1/\varepsilon$, whereas the fast variable $u(x,t)$ can be changed significantly in a much shorter time. Therefore, at the wave front [curve l_{AB} in Fig. 1(a)] and wave back [curve l_{CD} in Fig. 1(a)] of the propagating pulse, the values of the slow variable v, in the zero approximation on ε, can be taken as constants, where v_f corresponds to the wave front and v_b corresponds to the wave back.

These assumptions can be applied to system (2) if Δt is shorter than the excitation time T_{AB} or the deexcitation time T_{CD} [Fig. 1(b)]. Then, after the impulse is turned off, the values of the slow variable v for the wave front and wave back are, in the zero approximation on ε, also constant. From the second equation in Eqs. (2), we find that

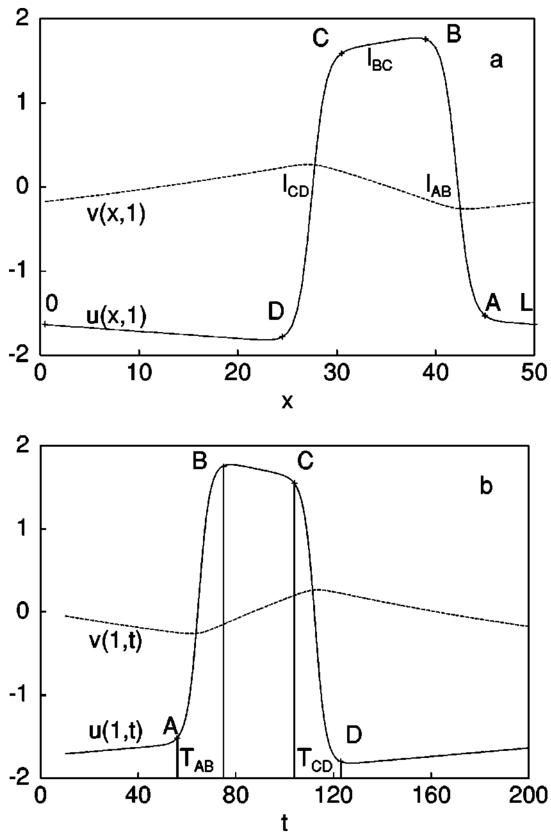

FIG. 1. (a) Spatial distributions $u(x,t=\text{const})$ and $v(x,t=\text{const})$. Curves l_{AB}, l_{BC}, and l_{CD}, respectively, correspond to the wave front, the excitation portion, and the wave back of the traveling wave. The pulse propagates to the right. (b) Time series of the fast $u(x=1,t)$ and slow $v(x=1,t)$ variables, where T_{AB} is the excitation time and T_{CD} is the deexcitation time.

$$v_{f,b}^{\text{imp}}=v_{f,b}+E, \tag{4}$$

where $E\equiv E_0\Delta t$. Therefore, the model describing the evolution of the wave front and wave back can be rewritten as

$$\frac{\partial u}{\partial t}=u-\frac{u^3}{3}-v_{f,b}^{\text{imp}}+D\frac{\partial^2 u}{\partial x^2}. \tag{5}$$

At a constant value of $v_{f,b}^{\text{imp}}$, Eq. (5) can have three spatially homogeneous steady states with coordinates u_1, u_2, and u_3, defined by the equation

$$\hat{f}(u)=u-u^3/3-v_{f,b}^{\text{imp}}=0. \tag{6}$$

Let $u_1<u_2<u_3$. Then two of these steady states, u_1 and u_3, are linearly stable, and u_2 is linearly unstable. Therefore, Eq. (5) describes two ways of switching: (1) from u_1 to u_3, which corresponds to the wave front; and (2) from u_3 to u_1, which corresponds to the wave back. These switching waves, which propagate with velocities $c_{f,b}$, are the solution of the equation

$$-c_{f,b}\frac{du}{d\xi}=u-\frac{u^3}{3}-v_{f,b}^{\text{imp}}+D\frac{d^2u}{d\xi^2}, \tag{7}$$

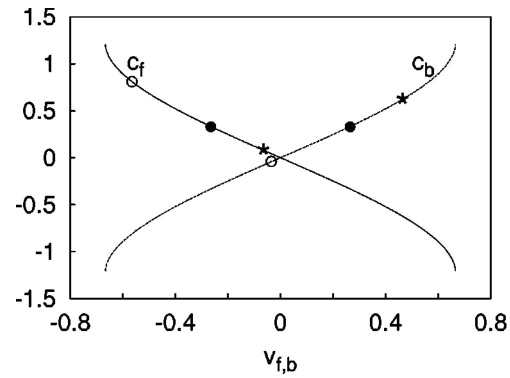

FIG. 2. Velocities of the pulse wave front c_f and wave back c_b vs $v_{f,b}$. For the unperturbed pulse, we have $v_b=-v_f\approx 0.265$, which leads to $c_b=c_f\approx 0.33$. The solid circles (●) denote the values for the unperturbed pulse. After the application of the impulse, the values for $v_{f,b}$ change and, as a consequence, the values for $c_{f,b}$ change. The open circles (○) and stars ($\star$), respectively, denote the values for the perturbed pulse at $E=-0.3$ and $E=0.2$.

where $\xi=x-c_{f,b}t$, where c_f is the wave front velocity and c_b the wave back velocity.

For the special case of a nonlinear cubic function $\hat{f}(u)$, it is possible to find an analytical solution to the above system. In such a case, the wave front and wave back velocities are defined as $c_{f,b}=\mp\sqrt{D/6}(u_1+u_3-2u_2)$, or, taking into account the solutions $u_i=u_i(v_{f,b}^{\text{imp}})$ of Eq. (6), as

$$c_{f,b}=\mp\sqrt{6D}\cos\frac{\arccos\left(-\frac{3}{2}v_{f,b}^{\text{imp}}\right)+\pi}{3}. \tag{8}$$

The wave front and wave back velocities versus $v_{f,b}$ are shown in Fig. 2. For the unperturbed, linearly stable propagating pulse, the velocity of the wave front and wave back must be equal. After the application of the impulse, the values of the slow variable at the wave front and wave back are changed to $v_{f,b}^{\text{imp}}$, as defined by Eq. (4). These changes lead to changes in the wave front and wave back velocities, as defined by Eq. (8). Subsequently, the propagation of the wave front and wave back with different velocities leads to shrinkage or expansion of the pulse width. If the amplitude and time duration of the impulse are sufficiently large, then the propagating pulse collapses and disappears. Positive values of E_0 destabilize the traveling wave by decreasing the pulse width (an effect we call ''pulse deexcitation''), whereas negative values of E_0 destabilize the wave by increasing the pulse width (an effect we call ''pulse excitation'').

Figure 3 shows numerical results for different values of E_0 with $\Delta t=0.5$. In all plots, the external impulse is applied at $t=0$. Plots of the time evolution of the distribution of the fast variable $u(x,0)$ of the propagating pulse are shown. In all plots, the dark color denotes excited regions, and the light color denotes unexcited regions. Figures 3(a)–3(c) correspond to pulse excitation and Figs. 3(d) and 3(e) correspond to pulse deexcitation.

Unsuccessful suppression is presented in Figs. 3(c) and 3(d). In these cases, the impulse $E\equiv E_0\Delta t$ was not sufficient to destabilize the initial propagating pulse. For a given Δt, E_0 must be larger than some critical value to suppress the

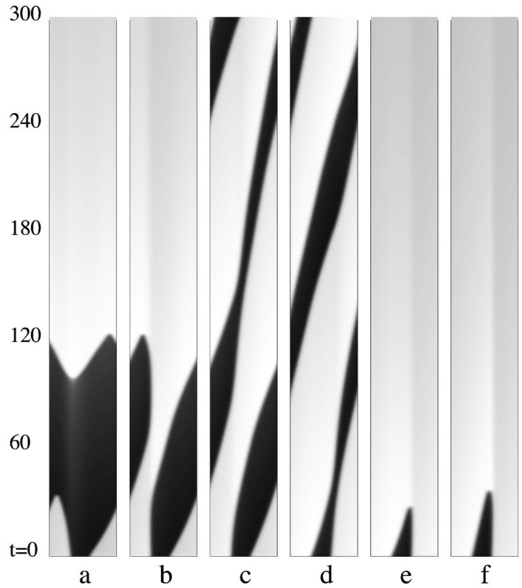

FIG. 3. Successful [(a), (b), (e), and (f)] and unsuccessful [(c) and (d)] suppression of a propagating pulse. Parameters: $\beta=0.7$, $\gamma=0.5$, $\varepsilon=0.005$, $\Delta t=0.5$, $E_0=-0.8$ (a), -0.6 (b), -0.4 (c), 0.3 (d), 0.4 (e), and 0.42 (f). In cases (a)–(e), the impulse is applied to the entire region. In case (f), the impulse is only applied to the pulse wave front.

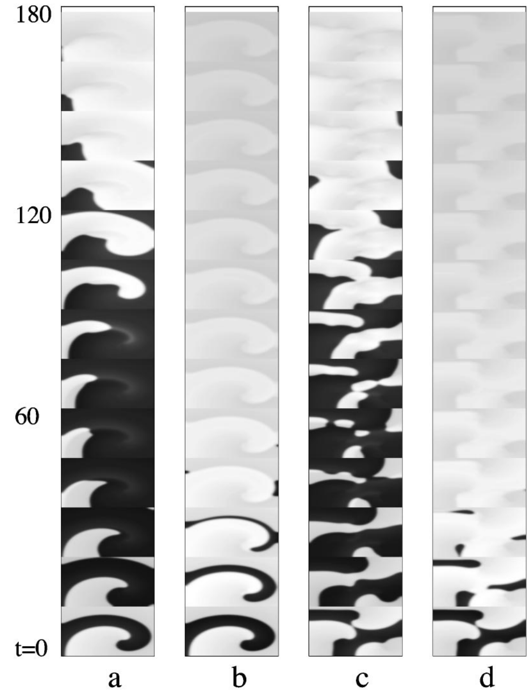

FIG. 5. Suppression of spiral waves and spiral-wave chaos. Snapshots of the spatial distribution of the variable $u(x,y)$ at different times. Parameters: $\beta=0.7$, $\gamma=0.5$, $\varepsilon=0.005$, $\Delta t=0.5$, $E_0=-0.84$ [(a) and (c)], and 0.64 [(b) and (d)].

traveling wave. Examples of successful suppression are presented in Figs. 3(a), 3(b), 3(e), and 3(f). In Fig. 3(a) (pulse excitation), following the application of the impulse, the full region becomes excited, which causes the medium to shift to the steady state. In Fig. 3(b) (also pulse excitation), the excitation does not propagate throughout the entire region. Here the propagating pulse initially widens, then shrinks, leading to a quenching of the pulse. From numerical analyses, we found that if $\Delta t \leqslant T_{AB,CD}$, then the critical value $E_{\rm cr}$ is practically the same for all Δt and E_0, and approximately equal to -0.241 for $L=50$ and -0.406 for $L=100$.

In the case of pulse deexcitation [Fig. 3(e)], the propagating pulse is suppressed if the impulse acts to decrease the width (Fig. 4) of the pulse to such a size that it cannot survive. Specifically, excitation is suppressed when the wave back reaches the wave front. For pulse deexcitation, we determined numerically that $E_{\rm cr}\approx 0.163$ for $L=50$ and $E_{\rm cr}\approx 0.286$ for $L=100$. In this case, the analytical estimation of $E_{\rm cr}$ can be obtained. If the sign of the wave front velocity c_f is reversed, then the pulse must collapse and disappear [16]. In our examples, $\hat{E}_{\rm cr}=-v_f\approx 0.265$ for $L=50$, and $\hat{E}_{\rm cr}=-v_f\approx 0.350$ for $L=100$.

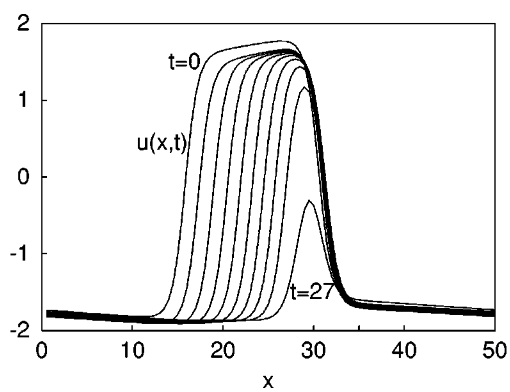

FIG. 4. Spatial distributions $u(x,t=n\tau)$ of the traveling pulse, for $n=0,\ldots,9$ and $\tau=3.0$, following the application of an impulse of duration $\Delta t=0.5$ and amplitude $E_0=0.4$. After the application of the impulse, the wave back propagates with essentially constant velocity $c_b=0.62$, whereas the wave front essentially does not propagate ($c_f=0.08$).

Note that it is not necessary to apply the impulse to the entire medium. The impulse could be applied in many different ways, including directly at the site of the wave front and wave back [e.g., see Fig. 3(f)]. The crucial factor is that the wave front and wave back velocities need to be changed a sufficient amount (see Fig. 2).

Now we consider using the same type of impulse control [Eq. (3)] to suppress a single spiral wave and spiral-wave chaos in a two-dimensional (100×100) model [in system (1), $\nabla^2 u=\partial^2 u/\partial x^2+\partial^2 u/\partial y^2$] with free boundary conditions. The spiral wave has its own characteristic width which is approximately constant far away from the core and the boundary of the medium. This width coincides with the width of a single pulse propagating in a one-dimensional medium at the same parameter values. Note that the movement of a spiral wave can be considered to consist of two parts: motion of the free end of the spiral, and motion of the front and back of the spiral wave [14]. Therefore, the action of an applied impulse on a single spiral wave or on an ensemble of interacting spiral waves should lead to effects similar to those we observed for pulse propagation in one-dimensional media. In particular, we found that the impulse can act to change the propagation speeds of the spiral front and spiral back of each wave, which can decrease or increase the width of the spiral. This can lead to the destabilization of the spiral wave and its annihilation.

Results for successful spiral-wave suppression with pulse excitation and deexcitation are presented in Figs. 5(a) and 5(b), respectively. In the case of pulse excitation [Fig. 5(a)], following the application of the impulse, the excited region (dark color) begins to increase, and expands into the larger portion of the medium. This occurs because the speed of the spiral front increases and the speed of the spiral back decreases. The spatial pattern resulting from this excitation expansion is unstable in the frame of the unperturbed equation (1), and collapses to the rest state. In the case of pulse deexcitation [Fig. 5(b)], the excited region decreases following the application of the impulse, and excitations are annihilated when the wave front and wave back reach one another.

We also performed numerical experiments on spiral-wave chaos [Figs. 5(c) and 5(d)] at the same parameter values as used above for a single spiral wave. We found that the associated traveling waves could be annihilated with impulses that were similar to those used to suppress single spiral waves. Thus the success of our suppression strategy appears to be more or less independent of the complexity of the spatial pattern. The critical feature is the width of the traveling wave.

Importantly, the amplitude and time duration of the impulse needed for wave suppression are relatively small. For example, for the parameter values we considered at $L=50$, an impulse of power $E_{\mathrm{cr}}\approx 0.93$ [17] is needed to depolarize each partial element of the medium. This value is approximately six times larger than that needed for pulse deexcitation, and approximately four times larger than that needed for pulse excitation via our proposed method.

Note that defibrillation procedures typically involve the application of relatively large external stimuli, that correspond to perturbations to the fast variable of excitable media models [6]. These perturbations result in the depolarization of most, if not all, of the media, which can serve to suppress existing traveling waves. Our proposed method, in contrast, operates on a slow variable and involves the application of weak impulses, as noted above. It may be possible to implement our method in cardiac tissue through the impulsive (pulsatile) application of drugs that influence slow currents in cardiac cells.

We thank M. Dolnik, A. Garfinkel, T. Kaper, D. Mar, A. Pikovsky, M. Spano, and A.M. Zhabotinsky for useful discussions. This work was supported by the U.S. Department of Energy and the National Science Foundation.

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[17] In this case, the critical value $E_{\mathrm{cr}}=|v_{\mathrm{max,min}}|+|v_{f,b}|\approx 0.666+0.265$, where $v_{\mathrm{max,min}}$ are the maximal and minimal values of the slow variable of the nullcline $f(u,v)=0$.